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Differential Operators and Primary Ideals

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0. INTRODUCTION

Let A be a (commutative) Noetherian ring containing a field k. Let $P \subset A$ be a prime ideal and let K be the field of fractions of A/P. The main result of this paper states that every P-primary ideal of A is the set of zeros of some finite dimensional *coalgebra* of differential operators from A to K. Actually, we make an additional finiteness assumption. If A' is the localization of A at P, we assume there is a field k', $k \subset k' \subset A'$, with $K \otimes_{k'} A'$ also Noetherian. This is perhaps more general than assuming K finitely generated over k'. In any event, there is no need to assume A finitely generated over a field. For example, A could be a quotient of a power series ring.

The differential operators $\mathrm{Diff}^n_{A/k}(A, K)$ form a subspace of the k-linear (but not necessarily A-linear) maps $\mathrm{Hom}_k(A, K)$. The precise definition is given in Section 1. We have $\mathrm{Diff}^n_{A/k}(A, K) = \mathrm{Diff}^n_{A'/k}(A', K) \supset \mathrm{Diff}^n_{A'/k'}(A', K)$. If $K \otimes_{k'} A'$ is Noetherian, then $\mathscr{D} = \mathrm{Diff}^n_{A'/k'}(A', K)$ is a finite dimensional coalgebra over K. Roughly, this means given $D \in \mathrm{Diff}^n_{A'/k'}(A', K)$, there is a formula for all $a, b \in A'$, $D(a \cdot b) = \Sigma D'(a)\, D''(b)$, for suitable D', $D'' \in \mathrm{Diff}^n_{A'/k'}(A', K)$. It is relatively easy to prove that if $\mathscr{E} \subset \mathscr{D}$ is a non-zero subcoalgebra, then $Q(\mathscr{E}) = \{a \in A \mid Da = 0 \text{ all } D \in \mathscr{E}\}$ is a P-primary ideal, containing P^{n+1}. The more interesting part of the theorem is the converse, that there are indeed enough subcoalgebras of operators to detect all primary ideals.

If, in the notation above, K is a separable algebraic extension of $k' \subset A'$, then, in fact, there is a bijective correspondence between subcoalgebras of $\mathrm{Diff}^n_{A'/k'}(A', K)$ and P-primary ideals containing P^{n+1}. If the characteristic is $p > 0$ and the degree of inseparability of K over k' is finite, say p^e, then we still get a bijective correspondence between P-primary ideals containing P^{n+1} and subcoalgebras of $\mathrm{Diff}^n_{A'/k'}(A', K)$, if the symbols are suitably interpreted. It is necessary to replace the obvious A'-module structure on A' and K by the structure induced by the p^e power map on A', that is, $a \circ x = a^{p^e} x$.

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If K is separably generated, but not algebraic over the ground field k, then $\mathrm{Diff}^n_{A/k}(A, K)$ has more subcoalgebras than A has P-primary ideals containing P^{n+1}. In fact, if Q is a P-primary ideal, let $Q' = QA'$. Then the set of all operators in $\mathrm{Diff}^n_{A/k}(A, K) = \mathrm{Diff}^n_{A'/k}(A', K)$ which vanish on Q' is exactly the natural subcoalgebra $\mathrm{Diff}^n_{(A'/Q')/k}(A'/Q', K)$. In particular, we have $\mathrm{Diff}^n_{K/k}(K, K) \subset \mathrm{Diff}^n_{A'/k}(A', K)$. If $k' \subset A'$ is chosen with K algebraic over k' then there is a natural map, which preserves the coproduct structures,

$$\sum_{i+j=n} \mathrm{Diff}^i_{A'/k'}(A', K) \otimes_K \mathrm{Diff}^j_{K/k}(K, K) \to \mathrm{Diff}^n_{A'/k}(A', K) = \mathrm{Diff}^n_{A/k}(A, K) \tag{0.1}$$

where the sum on the left is taken inside $\mathrm{Diff}^n_{A'/k'}(A', K) \otimes_K \mathrm{Diff}^n_{K/k}(K, K)$. The map (0.1) on generator $D_1 \otimes D_2$ is defined simply by composition of operators $D_2 \circ D_1 \colon A' \to K \to K$. (It is necessary to be careful about the meaning of $\mathrm{Diff}^j_{K/k}(K, K)$ as K-module, here. Scalars act inside operators, $x \circ D(y) = D(xy)$, whereas in the other two terms in (0.1) K-scalars act outside the operators.) We conjecture that (0.1) is always an isomorphism of K-vector spaces. (Although the map (0.1) is compatible with coproducts, it is not correct to refer to (0.1) as a coalgebra map because of the twisted K-module structure on $\mathrm{Diff}^j_{K/k}(K, K)$ needed to define (0.1).) We will prove (0.1) is an isomorphism for algebras of finite type over a field.

If $Q \supset P^{n+1}$, then $\mathscr{E}^i = \mathrm{Diff}^i_{(A'/Q')/k'}(A'/Q', K)$ is the canonical subcoalgebra of $\mathrm{Diff}^i_{A'/k'}(A', K)$ vanishing on Q'. If (0.1) is an isomorphism with A' replaced by A'/Q', then the largest subcoalgebra $\mathscr{E}'$ of $\mathrm{Diff}^n_{A'/k}(A', K)$ with zero set Q' is characterized as $\mathscr{E}' \cong \sum_{i+j=n} \mathscr{E}^i \otimes_K \mathrm{Diff}^j_{K/k}(K, K)$.

The geometric picture of these results is roughly as follows. We think of A as the coordinate ring of a variety V over k and A/P as the coordinate ring of a subvariety $W \subset V$. There is always a natural splitting $\mathrm{Diff}^n_{A/k}(A, K) = K \oplus \mathrm{Der}^n_{A/k}(A, K)$, where $\mathrm{Der}^n_{A/k}(A, K)$ consists of the operators D with $D(1) = 0$. We think of $D \in \mathrm{Der}^n_{A/k}(A, K)/\mathrm{Der}^{n-1}_{A/k}(A, K)$ as an nth order symmetric tangent tensor field, assigning to (almost all) points p of W a vector $D(p) \in S^n(T(V)_p)$. Thus if $n = 1$, D is a vector field, a sort of infinitesimal deformation of W in V. If $f \in A$ is a function, we think of Df as a derivative of f along the field. The operators $\mathrm{Der}^n_{K/k}(K, K)$ correspond to fields tangent to W in V. It is clear that derivatives along such W-tangent fields will measure nothing new about the functions $f \in P$ which already vanish identically on W.

If W is non-singular in V, then the sequence of bundles over W, $0 \to T(W) \to T(V) \to T(V)/T(W) \to 0$ is canonical and does split, but does not split naturally. The choice of subfield k', $k \subset k' \subset A'$, with K algebraic over k' can be interpreted as a kind of splitting of these vector bundles, at least generically, along W. We think of k' as the function field of a variety S, with a projection $V \to S$. The dimensions of S and W are the same and generically W is transverse in V to the fibres of $V \to S$. Thus, for example, the operators $D \in \mathrm{Der}^1_{A'/k'}(A', K)$ correspond to deformations of W in V above S. If W is a singular subvariety, this intuition is still useful, even though there is no normal bundle.

It seems to me that this characterization of primary ideals makes rather precise the well-known fact that rings with nilpotent elements arise in algebraic geometry in *relative* situations, to keep track of certain normal derivatives. For example, in intersections $W_1 \cap W_2$, and in parametrized families of varieties (fibres of a morphism $V \to S$) nilpotent elements arise because the geometry of these situations gives information not only about functions on certain subvarieties, but also about normal derivatives.

If $Q \subset A$ is a P-primary ideal, corresponding to a coalgebra of differential operators $\mathscr{E}$, we think of an element $f \in A/Q$ as defining not only the function f on W, but, in fact, the "tuple" of functions $\{Df\}_{D\in\mathscr{E}}$ on W. The product of two elements f, g becomes the tuple $\{D(fg)\}_{D\in\mathscr{E}} = \{\Sigma D'fD''g\}$, where $\Delta D = \Sigma D' \otimes D''$ is the diagonal formula in the coalgebra $\mathscr{E}$.

This picture gives one some geometric feeling for primary decomposition of ideals in Noetherian rings. The primary components of the isolated primes are well defined since in a geometric situation there is no ambiguity about which differential operators must vanish along the irreducible, isolated components of a subvariety. However, on the embedded components, there is ambiguity about choosing the coalgebra of operators in directions already tangent to other components. This is why the embedded primary components are not well defined. One can see, however, why the intersection of primary components belonging to an isolated *set* of primes should still be well defined.

There is nothing really new involved in the proof of our results. The material on differential operators is taken directly from papers of Heynemann and Sweedler [1] and Nakai [2]. The results on primary ideals which are used are exactly the results central to the fundamental theorem of Max Noether and the Hentzelt Nullstellensatz. (See for example [van der Waerden, Modern Algebra, Vol. II, Chapter XIII, Sections 96–97].) That is, one describes necessary and sufficient conditions for a polynomial f to belong to a primary ideal Q with radical prime $P \subset k[X_1 \cdots X_s]$, by first extending the ground field to reduce to the case of a zero dimensional primary ideal. This is our choice of k', $k \subset k' \subset A'$. Then one formulates linear conditions on the image of f in the finite dimensional vector space $k[X_1 \cdots X_s]/P^{n+1}$, where $Q \supset P^{n+1}$. These linear conditions are just that f should belong to the ideal Q/P^{n+1} in this finite dimensional algebra, which is equivalent to f being annihilated by an appropriate subcoalgebra of the dual coalgebra. This dual coalgebra is then identified with the appropriate nth order differential operators. Nonetheless, the precise formulation of the relations between primary ideals and zeros of differential operators does not seem to be in the literature.

Actually, we prove a more general result about primary submodules of a finitely generated module over a Noetherian ring with little extra work. If M is the A-module, we show that the P-primary submodules $L \subset M$ correspond to zeros of subcomodules of the differential operators $\mathrm{Diff}^n_{A/k}(M, K(M))$. Here $K(M) = K \otimes_A M$ and $\mathrm{Diff}^n_{A/k}(M, K(M))$ is naturally a finite dimensional comodule over the coalgebra $\mathrm{Diff}^n_{A/k}(A, K)$, at least assuming $K \otimes_k A$ is Noetherian. The generalization to modules is related to the fact that the natural domains of differential operators in topology are the modules of sections of vector bundles.

The paper is organized as follows. Basic definitions and the main results are stated in Section 1. Properties of differential operators are listed in Section 2. The main theorem for primary ideals is proved in Section 3, and the separable algebraic case is discussed in more detail in Section 4. In Section 5 we extend the main theorem to primary submodules. In Section 6 we clarify the role played by the choice of intermediate field $k \subset k' \subset A'$ by bringing in $\mathrm{Diff}^n_{A/k}(K, K)$. Finally, in Section 7 we suggest some natural extensions of the results and potential connections with other fields.

1. Statement of Results

Let A be a commutative ring with unit over a ground ring k. Let M, N be A-modules. (All modules and ring homomorphisms will be unitary.) The A-homomorphisms $\mathrm{Hom}_A(M, N)$ are in general a rather small submodule of the k-homomorphisms $\mathrm{Hom}_k(M, N)$. We regard $\mathrm{Hom}_k(M, N)$ as an $A \otimes_k A$ module by $(a \otimes b)\, u(m) = au(bm)$. In algebra and topology it is frequently useful to study modules between $\mathrm{Hom}_k(M, N)$ and $\mathrm{Hom}_A(M, N)$, defined by specific formulas for the deviation from A-linearity. This deviation is most naturally handled by using the bracket notation $[u, a](m) = u(am) - au(m)$, for $u \in \mathrm{Hom}_k(M, N)$, $a \in A$, $m \in N$. In particular, the nth *order differential operators* $\mathrm{Diff}^n_{A/k}(M, N)$ are defined inductively by

$$\mathrm{Diff}^{-1}_{A/k}(M, N) = (0)$$

$$\mathrm{Diff}^{i+1}_{A/k}(M, N) = \{u \mid [u, a] \in \mathrm{Diff}^i_{A/k}(M, N), \text{ all } a \in A\}.$$

In particular, $\mathrm{Diff}^0_{A/k}(M, N) = \mathrm{Hom}_A(M, N)$. If $I \subset A \otimes_k A$ denotes the kernel of the product map $\mu: A \otimes_k A \to A$, then I is generated by $\{a \otimes 1 - 1 \otimes a\}$ and $\mathrm{Diff}^0_{A/k}(M, N)$ is the $A \otimes_k A$ submodule of $\mathrm{Hom}_k(M, N)$ annihilated by I. An easy induction gives the following.

PROPOSITION 1.1. *$\mathrm{Diff}^n_{A/k}(M, N)$ is the $A \otimes_k A$ submodule of $\mathrm{Hom}_k(M, N)$ annihilated by I^{n+1}.*

We clearly have $0 \subset \mathrm{Diff}^0_{A/k}(M, N) \subset \mathrm{Diff}^1_{A/k}(M, N) \subset \cdots$. For fixed $a, b, c \in A$ and $u \in \mathrm{Hom}_k(M, N)$ a short computation shows $[(a \otimes b)u, c] = (a \otimes b)[u, c]$. Thus the maps $[\ , c]$: $\mathrm{Diff}^n_{A/k}(M, N) \to \mathrm{Diff}^{n-1}_{A/k}(M, N)$ are $A \otimes_k A$-module maps.

Let us say a k-subspace $\mathscr{E} \subset \mathrm{Diff}^n_{A/k}(M, N)$ is *closed* if $D \in \mathscr{E}$, $a \in A$ implies $[D, a] \in \mathscr{E}$. Let

$$\mathscr{E}^i = \mathscr{E} \cap \mathrm{Diff}^i_{A/k}(M, N), \qquad 0 \leqslant i \leqslant n,$$

so that $(0) \subset \mathscr{E}^0 \subset \mathscr{E}^1 \subset \cdots \subset \mathscr{E}^n = \mathscr{E}$ is a filtering of $\mathscr{E}$.

PROPOSITION 1.2. *If $\mathscr{E} \neq (0)$ is any closed subspace of $\mathrm{Diff}^n_{A/k}(M, N)$, then $\mathscr{E}^0 \neq (0)$.*

Proof. Let i be the least degree for which $\mathscr{E}^i \neq (0)$, and choose $D \neq 0 \in \mathscr{E}^i$. Then $[D, a] = 0$ for all $a \in A$, which simply says $D \in \mathscr{E}^0$. ∎

PROPOSITION 1.3. *Suppose $P \subset A$ is a prime ideal and N is a torsion free module over the integral domain A/P. Suppose $\mathscr{E} \subset \mathrm{Diff}^n_{A/k}(M, N)$ is a closed subspace. Then the set $L = \{m \in M \mid Dm = 0 \text{ all } D \in \mathscr{E}\}$ is a P-primary A-submodule of M, and $P^{n+1}M \subset L$.*

Proof. Since $D(am) = aD(m) + [D, a](m)$, L is an A-submodule of M. Choose $D \neq 0 \in \mathscr{E}^0$, by 1 2. Then $DM \neq (0) \subset N$, hence L is a proper submodule of M.

Next suppose $a \in P$, $m \in M$, $D \in \mathrm{Diff}^n_{A/k}(M, N)$. Since $D(a^{n+1}m) = aD(a^n m) + [D, a](a^n m) = [D, a](a^n m)$, we conclude by induction on n that $D(P^{n+1}M) = 0$. Thus certainly $P^{n+1}M \subset L$.

Finally, suppose $am \in L$, $a \in A$, $m \in L$. We need to prove $m \in L$ or $a \in P$. If $m \notin L$, let $D \in \mathscr{E}^i$ have least degree i with $Dm \neq 0$. Then $0 = D(am) = aD(m) + [D, a](m) = aD(m) \in N$. Since N is torsion free over A/P, we must have $a \in P$. ∎

Our main goal is a converse to Proposition 1.3 under suitable hypotheses. Let K denote the field of fractions of A/P, and consider $\mathscr{D}^n = \mathrm{Diff}^n_{A/k}(A, K)$. We want to characterize *all* P-primary ideals of A as zeros of closed subspaces $\mathscr{E} \subset \mathscr{D}^n$, some n. At this point it is convenient to replace the notion of closed subspace by subcoalgebra. Specifically, we have a diagram induced by $\mu\colon A \otimes_k A \to A$

$$\begin{array}{ccc} & & \mathrm{Hom}_k(A, K) \otimes_K \mathrm{Hom}_k(A, K) \\ & & \downarrow \\ \mathrm{Hom}_k(A, K) & \xrightarrow{\mu^*} & \mathrm{Hom}_k(A \otimes_k A, K). \end{array}$$

$\mathrm{Hom}_k(A, K) \otimes_k \mathrm{Hom}_k(A, K)$ is a subspace of $\mathrm{Hom}_k(A \otimes_k A, K)$ and if $D \in \mathrm{Hom}_k(A, K)$, we can ask if μ^*D belongs to this subspace. If so, say $\mu^*D = \Sigma D' \otimes D''$, we say D has a diagonal formula. This means $D(ab) = \Sigma D'(a)D''(b)$, all $a, b \in A$. In general, D will not have a diagonal formula. Roughly it is necessary that D annihilate a subspace of A of "finite codimension over K." (This requires interpretation since A is not a K-module.) This point is treated somewhat carefully in [1], but as we only need special cases, we dispense with the general discussion.

PROPOSITION 1.4.

(a) *If $K \otimes_k A$ is Noetherian, then every $D \in \mathrm{Diff}^n_{A/k}(A, K)$ has a diagonal formula. $\mathscr{D}^n = \mathrm{Diff}^n_{A/k}(A, K)$ is a finite dimensional, filtered, cocommutative coalgebra over K. That is, we have a cocommutative coassociative diagonal $\Delta: \mathscr{D}^n \to (\mathscr{D} \otimes_k \mathscr{D})^n$, where $(\mathscr{D} \otimes_k \mathscr{D})^n = \sum_{i+j=n} \mathscr{D}^i \otimes_K \mathscr{D}^j$, with co-unit $\eta: \mathscr{D}^n \to K$ defined by $\eta(D) = D(1)$. If we write $\Delta D = 1 \otimes D + \sum D' \otimes D''$, where $1 \in \mathrm{Diff}^0_{A/k}(A, K) = K$ is the natural map $A \to K$, then $[D, a] = \sum D'(a)D''$ for all $a \in A$.*

(b) *$\mathrm{Diff}^n_{A/k}(A, K)$ is naturally isomorphic to the coalgebra*

$$\mathrm{Hom}_K\left(\frac{K \otimes_k A}{\mathfrak{M}^{n+1}}, K\right)$$

dual to the finite dimensional filtered algebra $K \otimes_k A/\mathfrak{M}^{n+1}$ over K, where $\mathfrak{M}$ is the maximal ideal which is the kernel of the product map $K \otimes_k A \to K$.

Proposition 1.4 will follow from results in Section 2. The point is, 1.4 allows us to talk about subcoalgebras $\mathscr{E} \subset \mathscr{D}^n$ instead of closed subspaces.

Our main theorem is the following.

We assume from now on that the ground ring k is a field.

PROPOSITION 1.5.

(a) *If A and $K \otimes_k A$ are Noetherian and $Q \subset A$ is a P-primary ideal, then for some n, there is a subcoalgebra $\mathscr{E} \subset \mathrm{Diff}^n_{A/k}(A, K)$ such that $Q = \{a \in A \mid Da = 0$ all $D \in \mathscr{E}\}$.*

(b) *If K is separable algebraic over k, then there is a bijective correspondence between P-primary ideals of A which contain P^{n+1} and subcoalgebras of $\mathrm{Diff}^n_{A/k}(A, K)$.*

Proposition 1.5 will be proved in Sections 3, 4.

Now let M be a finitely generated A-module. We consider $\mathrm{Diff}^n_{A/k}(M, K(M))$ where $K(M) = K \otimes_A M$ and prove in Section 5 the following result, converse to 1.3. We continue to assume that A and $K \otimes_k A$ are Noetherian.

PROPOSITION 1.6.

(a) *$\mathrm{Diff}^n_{A/k}(M, K(M))$ is a finite dimensional comodule over $\mathrm{Diff}^n_{A/k}(A, K)$, naturally isomorphic to*

$$\mathrm{Hom}_K\left(\frac{K \otimes_k M}{\mathfrak{M}^{n+1}(K \otimes_k M)}, K(M)\right),$$

where $\mathfrak{M} \subset K \otimes_k A$ is kernel $(K \otimes_k A \to K)$.

(b) *Every P-primary submodule* $L \subset M$ *occurs as the set of zeros of some subcomodule* $\mathscr{E} \subset \mathrm{Diff}^n_{A/k}(M, K(M))$, *for some n.*

(c) *If K is separable algebraic over k, then there is a bijective correspondence between subcomodules of* $\mathrm{Diff}^n_{A/k}(M, K(M))$ *and P-primary submodules of M which satisfy* $P^{n+1}M \subset L$.

Finally, in Section 6 we will study the pairing (0.1) (in a special case)

$$\sum_{i+j=n} \mathrm{Diff}^i_{A'/k'}(A', K) \otimes_K \mathrm{Diff}^j_{K/k}(K, K) \to \mathrm{Diff}^n_{A'/k}(A', K) = \mathrm{Diff}^n_{A/k}(A, K) \tag{0.1}$$

defined as the composition $D_2 \circ D_1$ on elements $D_1 \otimes D_2$. Here, $k \subset k' \subset A'$ are a pair of subfields of A'.

PROPOSITION 1.7. *If A is finitely generated over k,* $P \subset A$ *a prime ideal,* $A' = A_{(P)}$, *and* $k' \subset A'$ *a subfield with* $k \subset k'$ *and K separable algebraic over* k', *then the map* (0.1) *above is an isomorphism. Moreover, if* $D_1 \in \mathrm{Diff}^i_{A'/k'}(A', K)$ *has diagonal formula* $\Delta D_1 = \sum D_1' \otimes D_1''$ *and* $D_2 \in \mathrm{Diff}^j_{K/k}(K, K)$ *has a diagonal formula* $\Delta D_2 = \sum D_2' \otimes D_2''$, *then* $\Delta(D_2 \circ D_1) = \sum\sum D_2' \circ D_1' \otimes D_2'' \circ D_1''$.

Note that the last assertion is trivial since it simply amounts to the computation $D_2D_1(ab) = D_2(\sum D_1'(a)\, D_2''(b)) = \sum\sum D_2'D_1'(a)\, D_2''D_1''(b)$, all $a, b \in A'$.

2. BASIC PROPERTIES OF DIFFERENTIAL OPERATORS

In this section we establish the basic properties of differential operators which we will need. All the results are more or less taken directly from [1] or [2]. We make no attempt to give a definitive discussion. For example, we do not discuss the representability of the functors $\mathrm{Diff}^n_{A/k}(A, \ )$ and $\mathrm{Diff}^n_{A/k}(\ , N)$, nor do we discuss the bialgebra structure of $\mathrm{Diff}^n_{A/k}(M, M)$, even though these properties could be nicely related to the results of this paper.

The first result is a formula which we do not really use, but which can be taken as an alternate definition of differential operators and which is useful if one actually wants some computational understanding, supplementing our inductive approach.

PROPOSITION 2.1. $D \in \mathrm{Diff}^n_{A/k}(M, N)$ *if and only if for all* $a_0, \dots, a_n \in A$, $m \in M$, *we have*

$$D(a_0 \cdots a_n m) = \sum_{s=0}^{n} \sum_{i_0 < \cdots < i_s} (-1)^s a_{i_0} \cdots a_{i_s} D(a_0 \cdots \hat{a}_{i_0} \cdots \hat{a}_{i_s} \cdots a_n m).$$

Proof. We have $[D, a] = (1 \otimes a - a \otimes 1)D$, hence by Proposition 1.1 $D \in \mathrm{Diff}^n_{A/k}(M, N)$ if and only if for all $a_0, \ldots, a_n \in A$

$$0 = (a_0 \otimes 1 - 1 \otimes a_0) \cdots (a_n \otimes 1 - 1 \otimes a_n)D = \sum_{s=0}^{n+1} \sum_{i_1 < \cdots < i_s} (-1)^s (a_{i_1} \cdots a_{i_s}) \otimes (a_0 \cdots \hat{a}_{i_1} \cdots \hat{a}_{i_s} \cdots a_n)D$$

where the empty product ($s = 0$) is interpreted as 1. ∎

Remark. If $M = A$, it suffices to check 2.1 with $m = 1$. This is ultimately so because a map $D: A \to N$ is A-linear if and only if $D(a \cdot 1) = aD(1)$ for all $a \in A$.

PROPOSITION 2.2. *Suppose $S \subset A$ is a multiplicative set and suppose N is an A_S module. Then the restriction map* $\mathrm{Hom}_k(M_S, N) \to \mathrm{Hom}_k(M, N)$ *induces an isomorphism, compatible with brackets* $[D, a]$,

$$\mathrm{Diff}^n_{A_S/k}(M_S, N) \xrightarrow{\simeq} \mathrm{Diff}^n_{A/k}(M, N).$$

Proof. We must show that every $D \in \mathrm{Diff}^n_{A/k}(M, N)$ extends uniquely to an operator $D_S \in \mathrm{Diff}^n_{A_S/k}(M_S, N)$. If $n = 0$, this is clear, since $\mathrm{Hom}_{A_S}(M_S, N) = \mathrm{Hom}_A(M, N)$. If $n > 0$, we must have for $m \in M$, $s \in S$,

$$\begin{aligned} D(m) &= D_S(m) = D_S(s(m/s)) \\ &= sD_S(m/s) + [D_S, s](m/s) \\ &= sD_S(m/s) + [D, s]_S(m/s). \end{aligned}$$

Thus we define inductively $D_S(m/s) = (1/s)(D(m) - [D, s]_S(m/s))$ and check that this works. ∎

PROPOSITION 2.3. *Suppose $J \subset A$ is an ideal and suppose N is an A/J module. Then the inclusion* $\mathrm{Hom}_k(M/JM, N) \subset \mathrm{Hom}_k(M, N)$ *induces an inclusion, compatible with brackets,*

$$\mathrm{Diff}^n_{(A/J)/k}(M/JM, N) \subset \mathrm{Diff}^n_{A/k}(M, N).$$

Proof. The $A \otimes_k A$ module structure on $\mathrm{Hom}_k(M/JM, N)$ factors through $A \otimes_k A \to A/J \otimes_k A/J$ and Proposition 1.1 can be applied. ∎

PROPOSITION 2.4. *If $k \subset K$ is an extension of ground rings and N is a $K \otimes_k A$ module, then the isomorphism* $\mathrm{Hom}_k(M, N) \simeq \mathrm{Hom}_K(K \otimes_k M, N)$ *induces an isomorphism* $\mathrm{Diff}^n_{A/k}(M, N) \simeq \mathrm{Diff}^n_{K \otimes_k A/K}(K \otimes_k M, N)$ *compatible with brackets.*

Proof. Let I_K denote the kernel of $(K \otimes_k A) \otimes_K (K \otimes_k A) \to K \otimes_k A$, and I the kernel of $A \otimes_k A \to A$. Under the identification $\mathrm{Hom}_k(M, N) = \mathrm{Hom}_K(K \otimes_k M, N)$, an operator D is annihilated by I^{n+1} if and only if it is annihilated by I_K^{n+1}. Now use Proposition 1.1. ▌

PROPOSITION 2.5. *Suppose $k' \subset A$ is a subring, with $k \subset k'$. Then the inclusion* $\mathrm{Hom}_{k'}(M, N) \subset \mathrm{Hom}_k(M, N)$ *induces an inclusion*

$$\mathrm{Diff}^n_{A/k'}(M, N) \subset \mathrm{Diff}^n_{A/k}(M, N),$$

compatible with brackets.

Proof. Trivial. ▌

PROPOSITION 2.6. *Suppose M, N, P are A-modules. Then the composition map* $\mathrm{Hom}_k(M, N) \times \mathrm{Hom}_k(N, P) \to \mathrm{Hom}_k(M, P)$ *induces a map*

$$\mathrm{Diff}^i_{A/k}(M, N) \otimes_A \mathrm{Diff}^j_{A/k}(N, P) \to \mathrm{Diff}^{i+j}_{A/k}(M, P),$$

where on the left A acts on the first factor via $1 \otimes A$ (N-coordinate) and on the second factor via $A \otimes 1$ (also the N-coordinate).

Proof. If $a \in A$, $D_1 \in \mathrm{Hom}_k(M, N)$, $D_2 \in \mathrm{Hom}_k(N, P)$, then $[D_2 \circ D_1, a] = D_2 \circ [D_1, a] + [D_2, a] \circ D_1$, so by induction, if D_1, D_2 are differential operators, then $D_2 \circ D_1$ will be a differential operator of the asserted order. Moreover, with the A-module structures indicated, $(aD_2) \circ D_1(m) = D_2(aD_1(m)) = D_2 \circ (aD_1)(m)$, $a \in A$, $m \in M$, hence the composition map extends to a well-defined map

$$\mathrm{Diff}^i_{A/k}(M, N) \otimes_A \mathrm{Diff}^j_{A/k}(N, P) \to \mathrm{Diff}^{i+j}_{A/k}(M, P).$$

Note this map is $A \otimes_k A$ linear if $A \otimes 1$ acts on the M-coordinate and $1 \otimes A$ on the P coordinate. ▌

PROPOSITION 2.7. *Suppose $J \subset A$ is an ideal such that the composition $k \to A \to A/J$ is an isomorphism, and suppose N is an $A/J = k$-module. Then*

$$\mathrm{Diff}^n_{A/k}(M, N) = \mathrm{Hom}(M/J^{n+1}M, N) \subset \mathrm{Hom}_k(M, N).$$

Proof. We have $(1 \otimes a - a \otimes 1)D(m) = D(am) - aD(m) = D(am - \epsilon(a)m)$, where $\epsilon: A \to A/J = k$ is the augmentation. (Since $aD(m) = \epsilon(a)\,D(m) \in N$.) Thus $(1 \otimes a - a \otimes 1)D = ((a - \epsilon(a))m)$. Since elements $a - \epsilon(a)$ generate J, we see by induction that $I^{n+1}D = 0$ if and only if $D(J^{n+1}M) = 0$. ▌

PROPOSITION 2.8. *Suppose the ground ring $k \subset A$ is a field and suppose $x \in A$ is separable algebraic over k. If $D \in \mathrm{Diff}^n_{A/k}(M, N)$, then $[D, x] = 0$. That is, D is $k(x)$-linear.*

Proof. Let $0 = f(x) = x^k + a_1x^{k-1} + \cdots + a_k$ be the minimal polynomial for x, $a_i \in k$. Then $f'(x) \neq 0$, and, in fact, $f'(x) \in k[x]$ is a unit in A. In $A \otimes_k A$ we have $0 = f(x \otimes 1) = f(1 \otimes x)$. But also $f(x \otimes 1) - f(1 \otimes x) = f'(1 \otimes x)(x \otimes 1 - 1 \otimes x) + (\text{terms})(x \otimes 1 - 1 \otimes x)^2$. Thus $(x \otimes 1 - 1 \otimes x) \equiv 0$ modulo $(x \otimes 1 - 1 \otimes x)^2$ and, by induction, $(x \otimes 1 - 1 \otimes x) \equiv 0$ modulo $(x \otimes 1 - 1 \otimes x)^{n+1}$. It follows that $[D, x] = (1 \otimes x - x \otimes 1)D = 0$. ▮

The significance of Proposition 2.8 is that we can always replace k by its separable algebraic closure in A without changing $\mathrm{Diff}^n_{A/k}(M, N)$.

We will now establish Proposition 1.4. Recall we assume A is a Noetherian ring containing a field k. We have $P \subset A$ a prime ideal, and K the field of fractions of A/P. We also assume $K \otimes_k A$ is Noetherian.

By Proposition 2.4, $\mathrm{Diff}^n_{A/k}(A, K) \cong \mathrm{Diff}^n_{K \otimes_k A/K}(K \otimes_k A, K)$. But $K \otimes_k A$ is an augmented algebra over K, that is, the product map $\epsilon: K \otimes_k A \to K$, with kernel $\mathfrak{M}$, is an isomorphism on $K \otimes 1$. Thus by Proposition 2.7,

$$\mathrm{Diff}^n_{K \otimes_k A/K}(K \otimes_k A, K) \cong \mathrm{Hom}_K(K \otimes_k A/\mathfrak{M}^{n+1}, K).$$

Since $K \otimes_k A$ is Noetherian, $K \otimes_k A/\mathfrak{M}^{n+1}$ is a finite dimensional K-algebra, and its dual is a finite dimensional coalgebra over K. This proves Proposition 1.4(b). (In fact, the same argument, making use of Proposition 2.7, gives the analogous result Proposition 1.6(a) for modules.) Proposition 1.4(a) is essentially contained in Proposition 1.4(b).

There is one further point to be made concerning filtrations. The K-algebra $V = K \otimes_k A/\mathfrak{M}^{n+1}$ is filtered by $V \supset V_0 \supset V_1 \supset \cdots \supset V_n = 0$ where $V_i = \mathfrak{M}^{i+1}/\mathfrak{M}^{n+1}$, $i \geqslant 0$. The dual $V^* = \mathrm{Hom}_K(V, K)$ is thus also filtered by $0 = V^\perp \subset V_0{}^\perp \subset \cdots \subset V_n{}^\perp = V^*$, where $V_i{}^\perp$ are the homomorphisms which vanish on V_i. Thus $V_i{}^\perp = \mathrm{Diff}^i_{A/k}(A, K)$. Now, Proposition 1.4(a) asserts that V^* is a filtered coalgebra, that is, $\Delta V^* \subset \sum_{i+j=n} V_i{}^\perp \otimes_K V_j{}^\perp \subset V^* \otimes_K V^*$. What is completely obvious is that $\Delta V^* \subset V^* \otimes_K V^*$ annihilates

$$\sum_{r+s=n-1} V_r \otimes_K V_s \subset V \otimes_K V.$$

On the other hand, $\sum_{i+j=n} V_i{}^\perp \otimes_K V_j{}^\perp$ is the annihilator of $\bigcap_{i+j=n}(V_i \otimes_K V + V \otimes_K V_j)$. It is a little exercise that, in fact,

$$\sum_{r+j=n-1} V_r \otimes_K V_s = \bigcap_{i+j=n} (V_i \otimes_K V + V \otimes_K V_j) \subset V \otimes_K V.$$

3. Primary Ideals

Our goal in this section is the first part of the main Theorem, 1.5(a). We continue with the notation and assumptions of the previous section, concerning $P \subset A$ and $K \otimes_k A$.

Let A' be the local ring $A_{(p)}$ and $P' \subset A'$ the maximal ideal. By Proposition 2.2, $\mathrm{Diff}^n_{A/k}(A, K) = \mathrm{Diff}^n_{A'/k}(A', K)$. Also, the P-primary ideals in A correspond bijectively with all ideals in A' with radical P'. Thus in relating P-primary ideals and subcoalgebras of $\mathrm{Diff}^n_{A/k}(A, K)$, we may as well assume A is a local ring and P is its maximal ideal. We make this assumption for the remainder of this section.

Consider $K \otimes_k A$. The augmentation $K \otimes_k A \to K$ factors, $K \otimes_k A \to K \otimes_{k'} A \to K$, for any subring k', $k \subset k' \subset A$. Since A is a local ring, we can find a field $k \subset k' \subset A$, such that K is algebraic over k'. To see this, use Zorn's Lemma to choose a maximal field k' in A containing k. If $x \in A$ represents an element of K transcendental over k', then every non-zero element of $k'[x] \subset A$ is a unit in A. But this implies $k'(x) \subset A$, contradicting maximality of k'.

From 2.4, $\mathrm{Diff}^n_{A/k}(A, K) \cong \mathrm{Diff}^n_{K\otimes_k A}(K \otimes_k A, K)$, which we now know is a finite dimensional coalgebra over K. From 2.3 and 2.5, we have subcoalgebras

$$\begin{array}{ccc} \mathrm{Diff}^n_{A/k'}(A, K) & \subset & \mathrm{Diff}^n_{A/k}(A, K) \\ \| \wr & & \| \wr \\ \mathrm{Diff}^n_{K\otimes_{k'} A/K}(K \otimes_{k'} A, K) & \subset & \mathrm{Diff}^n_{K\otimes_k A/K}(K \otimes_k A, K). \end{array}$$

Thus, in order to prove 1.5(a), it suffices to prove that every P-primary ideal is the set of zeros of some subcoalgebra of $\mathrm{Diff}^n_{A/k'}(A, K)$, some n. That is, we may assume K algebraic over k.

On the other hand, in the "geometric case", the natural ground field k is, say, an algebraically closed field, and A (before localization) is an algebra of finite type over k, corresponding to a closed subvariety V of some affine space over k. The prime $P \subset A$ corresponds to an irreducible subvariety $W \subset V$ of some dimension (the transcendence degree of K over k). Extending the ground field $k \subset k'$ amounts to choosing a transcendence base in A for K over k, thus reducing the algebra to a zero dimensional situation. But we should still be very interested in $\mathrm{Diff}^n_{A/k}(A, K)$ because of the close relations with the geometry of the embedding $W \subset V$. Thus we will study the inclusion $\mathrm{Diff}^n_{A/k'}(A, K) \subset \mathrm{Diff}^n_{A/k}(A, K)$ in more detail in Section 6.

In any event, assuming for now that K is algebraic over k, consider the integral extension $A = 1 \otimes A \subset K \otimes_k A$. Let $\mathfrak{M} \subset K \otimes_k A$ be the kernel of $K \otimes_k A \to K$, as above, so that $P = \mathfrak{M} \cap A$. Since

$$\mathrm{Diff}^n_{A/k}(A, K) \cong \mathrm{Diff}^n_{K\otimes_k A/K}(K \otimes_k A, K) \cong \mathrm{Hom}_K(K \otimes_k A/\mathfrak{M}^{n+1}, K),$$

it is easy to see that coalgebras $\mathrm{Diff}^n_{A/k}(A, K)$ correspond bijectively to ideals in the Artinian ring $K \otimes_k A/\mathfrak{M}^{n+1}$ that is, to $\mathfrak{M}$-primary ideals $\mathfrak{N} \subset K \otimes_k A$ containing $\mathfrak{M}^{n+1}$. Thus to prove 1.5(a), it is only necessary to argue that if

$Q \subset A$ is a P-primary ideal, then there is an $\mathfrak{M}$-primary ideal $\mathfrak{N}$ with $Q = \mathfrak{N} \cap A$.

But this last is a well-known fact. For completeness we give the argument. Nothing is lost by replacing K by any finite extension $K' \supset K$ and replacing $\mathfrak{M}$ by $\mathfrak{M}'$, the kernel of $K' \otimes_k A \to K'$. That is, it clearly suffices to find an $\mathfrak{M}'$-primary ideal $\mathfrak{N}'$ with $\mathfrak{N}' \cap A = Q$. Thus we may assume that K contains all the conjugates (in some algebraic closure) of any given finite set of elements of K.

Any prime ideal $\mathfrak{M}_i$ of $K \otimes_k A$ containing $p^e = P(K \otimes_k A)$ is a maximal ideal, minimal over P. This follows, say, from the going up theorem, or (more elementary), from the fact that every element of $K \otimes_k A/P(K \otimes_k A) = K \otimes_k K$ is algebraic over $K \otimes 1$. Thus there are finitely many such $\mathfrak{M}_i$, say $1 \leqslant i \leqslant r$, with $\mathfrak{M}_1 = \mathfrak{M}$, and these will also be the only primes in $K \otimes_k A$ over $Q^e = Q(K \otimes_k A)$ for any P-primary ideal Q. We thus have a unique primary decomposition in $K \otimes_k A$, $Q^e = \mathfrak{N}_1 \cap \cdots \cap \mathfrak{N}_r$, with $\mathfrak{N}_i$ an $\mathfrak{M}_i$-primary ideal. We will prove $Q = \mathfrak{N}_1 \cap A$.

First, $Q = Q^e \cap A$ since A/Q injects into $K \otimes_k (A/Q) = K \otimes_k A/Q^e$. Thus $Q = \mathfrak{N}_1 \cap A$ will follow if we prove that $a \in \mathfrak{N}_1 \cap A$ implies $a \in \mathfrak{N}_i$, all i.

The primary components $\mathfrak{N}_i$ of Q^e are characterized as follows.

$$\mathfrak{N}_i = \{x \in K \otimes_k A \mid xs \in Q^e, \text{ some } s \notin \mathfrak{M}_i\}.$$

Since the $\mathfrak{M}_i$ contain P^e, the $\mathfrak{M}_i$ arise as kernels of homomorphisms $\sigma_i\colon K \otimes_k A \to K \otimes_k K \to L$ where L is the algebraic closure of K. We may assume $\sigma_i |_{1\otimes A} = 1\colon A \to L$ (the obvious map) say by an automorphism of L if necessary. We also denote by σ_i the restriction $\sigma_i |_{K\otimes 1}\colon K \to L$. Now suppose $a \in \mathfrak{N}_1 \cap A$, say $as \in Q^e$ with $s = \sum \alpha_j \otimes a_j \notin \mathfrak{M}_1 = \mathfrak{M}$. Then $\sum \alpha_j \bar{a}_j \neq 0 \in K$. where $\bar{a}_j \in K = A/P$ is the residue class of a_j. As discussed above, we may assume K contains all conjugates of the α_j in L. Also, if we write

$$as = \sum_m \sum_l (\beta_{lm} \otimes b_{lm})\, c_m$$

with $c_m \in Q$, we may assume K contains all the conjugates of the β_{lm}. Let $s_i = \sum \sigma_i^{-1}(\alpha_j) \otimes a_j$, so that $\sigma_i s_i = \sum \alpha_j \bar{a}_j \neq 0$. Thus $s_i \notin \mathfrak{M}_i$. But $as_i = \sum \sigma_i^{-1}(\alpha_j) \otimes a_j a = \sum_m \sum_l (\sigma_i^{-1}(\beta_{lm}) \otimes b_{lm}) c_m$ since these elements have the same image under the injection $\sigma_i \otimes 1\colon K \otimes_k A \to L \otimes_k A$. Thus $as_i \in Q^e$ and hence $a \in \mathfrak{N}_i$ as desired.

If K is finite over k, the argument is simpler. First, one replaces K by a normal extension of k. Then all the $\mathfrak{M}_i$ and the $\mathfrak{N}_i$ are actually conjugate in $K \otimes_k A$. ∎

This completes the proof of 1.5(a). We also see that if $Q \subset A$ is a P-primary ideal, $\mathfrak{M} \subset K \otimes_k A$ as above, then Q will be the set of zeros of a subcoalgebra $\mathscr{E} \subset \mathrm{Diff}^n_{A/k}(A, K)$ where n is such that the $\mathfrak{M}$-primary component $\mathfrak{N}$ of Q^e contains $\mathfrak{M}^{n+1}$.

Actually, a more precise statement can be made. If K is separably generated over the original ground field $k \subset A$ (that is, K separable algebraic over some subfield $k' \subset A$), then Q will be the zeros of a subcoalgebra $\mathscr{E} \subset \mathrm{Diff}^n_{A/k}(A, K)$ if $Q \supset P^{n+1}$. This is 1.5(b) and will be proved in the next section. If K has degree of separability p^e over some subfield $k' \subset A$ and $Q \supset P^{n+1}$, then Q will be the zeros of a subcoalgebra $\mathscr{E} \subset \mathrm{Diff}^{np^e}_{A/k}(A, K)$.

If the characteristic is p, then A and K can be given a new A-module structure by means of the p^e power map $A \to A$. Thus $a \circ x = a^{p^e}x$, $a \in A$, $x \in K$. Write A_e, K_e for these new A-module structures. If $D \in \mathrm{Hom}_k(A_e, K_e)$ and $a \in A$, then $[D, a]b = D(a^{p^e}b) - a^{p^e}D(b)$. Since $(a^{p^e} \otimes 1 - 1 \otimes a^{p^e}) = (a \otimes 1 - 1 \otimes a)^{p^e} \in A \otimes_k A$, we have immediately from 1.1 that $\mathrm{Diff}^{np^e}_{A/k}(A, K) \subset \mathrm{Diff}^n_{A/k}(A_e, K_e)$.

4. The Separable Case

In the notation and with the assumptions of the previous section, the contraction map from $\mathfrak{M}$-primary ideals of $K \otimes_k A$ to P-primary ideals of A is surjective, but not necessarily bijective. The ramification is due to inseparability if K is algebraic over k.

For example, suppose k is a field of characteristic p, $a \in k$, $a \notin k^p$. Let P be the maximal ideal $(X^p - a) \subset k[X] = A$. Then $K = k[\alpha]$ where $\alpha^p = a$. Also $K \otimes_k A = K[X]$ and $\mathfrak{M} = (X - \alpha) \subset K[X]$. Since $X^p - a = (X - \alpha)^p \in K[X]$, we see that above each P-primary ideal $(X^p - a)^j$ of $k[X]$ we have p $\mathfrak{M}$-primary ideals of $K[X]$, namely, $(X - \alpha)^{pj}$, $(X - \alpha)^{pj+1}, \ldots, (X - \alpha)^{p(j+1)-1}$. Since $\mathrm{Diff}^n_{k[X]/k}(k[X], K) = \mathrm{Diff}^n_{K[X]/K}(K[X], K) = \mathrm{Hom}_k(K[X]/(X - \alpha)^{n+1}, K)$, we also have the ramification in the surjective map from subcoalgebras of $\mathrm{Diff}^n_{k[X]/k}(k[X], K)$ to $(X^p - a)$-primary ideals of $k[X]$.

However, suppose A is a local ring, $P \subset A$ the maximal ideal, $k \subset A$ a subfield so that $K = A/P$ is separable algebraic over k. Assume $K \otimes_k A$ is Noetherian, with its canonical maximal ideal $\mathfrak{M}$ as before.

Proposition 4.1. *The natural map $A/P^{n+1} \cong K \otimes_k A/\mathfrak{M}^{n+1}$ is an isomorphism of rings.*

Note that it is then trivial that $\mathfrak{M}$-primary ideals in $K \otimes_k A$ correspond bijectively with P-primary ideals in A. Thus, combined with the discussion in the preceding section, Proposition 4.1 implies Proposition 1.5(b) in the case K separable algebraic over k.

Proof. Certainly if $n = 0$, $K = A/P \cong K \otimes_k A/\mathfrak{M} = K$ is an isomorphism. Using the exact sequences

$$0 \to P^i/P^{i+1} \to A/P^{i+1} \to A/P^i \to 0$$

and induction, it suffices to prove that the map $P^i/P^{i+1} \to \mathfrak{M}^i/\mathfrak{M}^{i+1}$ is an isomorphism of K vector spaces, for all i.

If $P/P^2 \to \mathfrak{M}/\mathfrak{M}^2$ is surjective, then $P^i/P^{i+1} \to \mathfrak{M}^i/\mathfrak{M}^{i+1}$ is surjective for all i. But $\mathfrak{M} \subset K \otimes_k A$ is generated by elements $\alpha \otimes 1 - 1 \otimes a$ where $a \in A$, $\alpha = \bar{a} \in K$. Let $0 = f(\alpha)$ be a separable polynomial for α over k. Then $f(a) \in P \subset A, f'(a) \notin P$ is a unit in A and we have the Taylor expansion

$$0 = f(\alpha \otimes 1) = f(1 \otimes a) + f'(1 \otimes a)(\alpha \otimes 1 - 1 \otimes a)$$
$$+ (\text{terms})(\alpha \otimes 1 - 1 \otimes a)^2.$$

Thus

$$\alpha \otimes 1 - 1 \otimes a = (-1/f'(1 \otimes a)) f(1 \otimes a) (\text{mod } \mathfrak{M}^2)$$

and therefore $P/P^2 \to \mathfrak{M}/\mathfrak{M}^2$ is surjective.

To show that $P^i/P^{i+1} \to \mathfrak{M}^i/\mathfrak{M}^{i+1}$ is injective, assume the kernel is Q. Again, we are free to extend K by adjoining conjugates of elements. Thus Q is a P-primary ideal, $Q \supset P^{i+1}$ and $Q^e = Q(K \otimes_k A) \subset \mathfrak{M}^{i+1}$. Let $\mathfrak{M} = \mathfrak{M}_1, \ldots, \mathfrak{M}_r$ be the primes of $K \otimes_k A$ above P. Since $Q^e \subset \mathfrak{M}^{i+1}$, we can deduce that $Q^e \subset \mathfrak{M}_j^{i+1}$, all j. This is proved easily under the assumption that K contains all conjugates of all elements occuring in a formula in $K \otimes_k A$, expressing the generators of Q as elements of $\mathfrak{M}^{i+1}$. We thus have $Q^e \subset \mathfrak{M}_1^{i+1} \cap \cdots \cap \mathfrak{M}_r^{i+1}$.

But $\mathfrak{M}_1^{i+1} \cap \cdots \cap \mathfrak{M}_r^{i+1}$ is precisely the primary decomposition of $(P^{i+1})^e$. That is, $\mathfrak{M}^{i+1}$ is the smallest $\mathfrak{M}$-primary ideal containing P^{i+1}. This follows from the surjectivity $P^{i+1} \to \mathfrak{M}^{i+1}/\mathfrak{M}^{i+2}$ and Nakayama's lemma applied to the formula $\mathfrak{M}^{i+1} = (P^{i+1})^e + \mathfrak{M} \cdot \mathfrak{M}^{i+1}$ (in the local ring $(K \otimes_k A)_{(\mathfrak{M})}$). For the other $\mathfrak{M}_j$ we use conjugate elements, to show $\mathfrak{M}_j^{i+1}$ is the smallest $\mathfrak{M}_j$-primary ideal containing P^{i+1}. Thus $Q \subset A \cap (P^{i+1})^e = P^{i+1}$, hence $P^i/P^{i+1} \to \mathfrak{M}^i/\mathfrak{M}^{i+1}$ is injective. This completes the proof of Proposition 4.1. ▮

From 4.1 A/P^{n+1} becomes a finite dimensional algebra over K. Now, A/P^{n+1} is also an algebra over k, finite dimensional if $| K : k | < \infty$, which we now assume. Thus $\mathrm{Hom}_k(A/P^{n+1}, k)$ is a coalgebra over k and, of course, the subcoalgebras will correspond bijectively with the primary ideals of A containing P^{n+1}. Since k is not an A-module, we do not interpret $\mathrm{Hom}_k(A/P^{n+1}, k)$ as differential operators from A to k, at least not directly. What then is the relation between the coalgebra $\mathrm{Hom}_k(A/P^{n+1}, k)$ over k and the coalgebra $\mathrm{Hom}_K(K \otimes_k A/\mathfrak{M}^{n+1}, K)$ over K?

Consider the trace, tr: $K \to k$, which is a non-zero functional in the separable case. The pairing $K \times K \to k$ defined by $\mathrm{tr}(xy)$ identifies K and $K^* = \mathrm{Hom}_k(K, k)$. More generally, if V is a finite dimensional K-vector space, then the trace defines by composition an *isomorphism* $\mathrm{Hom}_K(V, K) \quad \mathrm{Hom}_k(V, k)$ of k-vector spaces.

Write $V_K{}^* = \mathrm{Hom}_K(V, K)$, $V_k{}^* = \mathrm{Hom}_k(V, k)$. If V is an algebra over K, then $V_K{}^*$ is a coalgebra over K, $V_k{}^*$ is a coalgebra over k and we have an isomorphism tr: $V_K{}^* \simeq V_k{}^*$. We want to consider the compatibility of trace with the coproducts Δ_K: $V_K{}^* \to V_K{}^* \otimes_K V_K{}^*$ and Δ_k: $V_k{}^* \to V_k{}^* \otimes_k V_k{}^*$. Note that the natural map between the ranges of these coproducts is

$$V_k{}^* \otimes_k V_k{}^* \xrightarrow{\mathrm{tr}^{-1} \otimes \mathrm{tr}^{-1}} V_K{}^* \otimes_k V_K{}^* \xrightarrow{\pi} V_K{}^* \otimes_K V_K{}^*$$

where π is the obvious map induced by the identity on $V_K{}^* \times V_K{}^*$.

PROPOSITION 4.2. *If* $D \in V_K{}^*$ *and*

$$\Delta_k \operatorname{tr}(D) = \sum \operatorname{tr}(D') \otimes \operatorname{tr}(D'') \in V_k{}^* \otimes_k V_k{}^*,$$

then $\Delta_K D = \sum D' \otimes D'' \in V_K{}^* \otimes_K V_K{}^*$.

Proof. By the definition of the coproducts, this formula asserts that if D, D', $D'' \in \operatorname{Hom}_K(V, K)$ and if $\operatorname{tr} D(ab) = \sum \operatorname{tr} D'(a) \operatorname{tr} D''(b) \in k$ for all $a, b \in V$, then $D(ab) = \sum D'(a)\, D''(b) \in K$.

Choose a k-base of K, say $\{x_1 \cdots x_n\}$. Let $\{x_i{}^*\}$ be the dual base, relative to the non-degenerate form $\operatorname{tr}(xy)$. Thus if $y, z \in K$, $y = \sum \operatorname{tr}(x_i y) x_i{}^*$, $z = \sum \operatorname{tr}(x_j{}^* z) x_j$ and $\operatorname{tr}(yz) = \sum \operatorname{tr}(x_i y) \operatorname{tr}(x_i{}^* z)$. We also claim $\sum x_i x_i{}^* = 1$. To prove this it suffices to prove $\operatorname{tr}(y) = \operatorname{tr}(\sum y x_i x_i{}^*)$ for all $y \in K$. But $\operatorname{tr}(y)$ is by one definition the sum of the diagonal entries of a matrix over k representing the k-linear transformation y: $K \to K$. Since $y x_j{}^* = \sum_i \operatorname{tr}(y x_j{}^* x_i) x_i{}^*$, we have $\operatorname{tr}(y) = \sum_j \operatorname{tr}(y x_j{}^* x_j)$.

Now, we prove the desired formula $D(ab) = \sum D'(a)\, D''(b) \in K$, all $a, b \in V$. It suffices to prove $\operatorname{tr}(D(ab)y) = \sum \operatorname{tr}(D'(a)\, D''(b) y)$, all $y \in K$. But D is K-linear, so $D(ab)y = \sum x_i x_i{}^* D(ab) y = \sum D(x_i a \cdot x_i{}^* by)$ and $\operatorname{tr}(D(ab)y) = \sum\sum \operatorname{tr} D'(x_i a) \operatorname{tr} D''(x_i{}^* by)$. On the other hand, we observed in the preceding paragraph that $\operatorname{tr}(\sum D'(a)\, D''(b) y) = \sum\sum \operatorname{tr}(x_i\, D'(a)) \operatorname{tr}(x_1{}^*\, D''(b) y)$ and our formula follows since D' and D'' are also K-linear. ∎

5. PRIMARY SUBMODULES

In this section, we complete our discussion of primary submodules by proving Proposition 1.6. We continue the notation and assumptions of the previous sections.

If M is a finitely generated module over A, let $M' = M \otimes A' = M_{(P)}$, where $P \subset A$ is our prime ideal, $A' = A_{(P)}$. By Proposition 2.2,

$$\operatorname{Diff}^n_{A/k}(M, K(M)) \cong \operatorname{Diff}^n_{A'/k}(M', K(M)).$$

Also, it is easy to check that P-primary submodules of M correspond bijectively with $PA' = P'$-primary submodules of M'. It is further the case that a submodule $L' \subset M'$ is P'-primary if and only if $P^{n+1}M' \subset L'$ for some n, that is, we want all submodules of $M'/P^{n+1}M'$. Thus, we may as well assume A is a local ring with maximal ideal P, and we are trying to show that every P-primary submodule of M arises as the zeros of some subcomodule of differential operators $\mathscr{E} \subset \mathrm{Diff}^n_{A/k}(M, K(M))$ over the coalgebra $\mathrm{Diff}^n_{A/k}(A, K)$, some n.

From Section 2, we have isomorphisms

$$\mathrm{Diff}^n_{A/k}(A, K) \cong \mathrm{Diff}^n_{K\otimes_k A/K}(K \otimes_k A, K) \cong \mathrm{Hom}_K(K \otimes_k A/\mathfrak{M}^{n+1}, K)$$

and

$$\begin{aligned}\mathrm{Diff}^n_{A/k}(M, K(M)) &\subseteq \mathrm{Diff}^n_{K\otimes_k A/K}(K \otimes_k M, K(M)) \\ &\cong \mathrm{Hom}_K(K \otimes_k M/\mathfrak{M}^{n+1}(K \otimes_k M), K(M)).\end{aligned}$$

By our finiteness assumptions, $K \otimes_k M/\mathfrak{M}^{n+1}(K \otimes_k M)$ is a finite dimensional K-vector space, in fact, a module over $K \otimes_k A/\mathfrak{M}^{n+1}$. Also $K(M) = M/PM$ is a finite dimensional K-vector space. Thus the natural map

$$\begin{aligned}&\mathrm{Hom}_K(K \otimes_k A/\mathfrak{M}^{n+1}, K) \otimes_K \mathrm{Hom}_K(K \otimes_k M/\mathfrak{M}^{n+1}(K \otimes_k M), K(M)) \\ &\quad \to \mathrm{Hom}_K((K \otimes_k A/\mathfrak{M}^{n+1}) \otimes_K (K \otimes_k M/\mathfrak{M}^{n+1}(K \otimes_k M)), K(M))\end{aligned}$$

is bijective. In this way $\mathrm{Diff}^n_{K\otimes_k A/K}(K \otimes_k M, K(M))$ is a finite dimensional comodule over the coalgebra $\mathrm{Diff}^n_{K\otimes_k A/K}(K \otimes_k A, K)$, which gives us Prop. 1.6(a). Also, the $K \otimes_k A$ submodules of $K \otimes_k M/\mathfrak{M}^{n+1}(K \otimes_k M)$ correspond bijectively with subcomodules of $\mathrm{Diff}^n_{K\otimes_k A/K}(K \otimes_k M, K(M))$.

Every $\mathfrak{M}$-primary $K \otimes_k A$-submodule of $K \otimes_k M$ determines by contraction a P-primary A-submodule of M. We assert this contraction map is *surjective*. This will prove Prop. 1.6(b), using the above paragraph.

As in Section 3, we can extend the ground field $k \subset k' \subset A$, so that K is algebraic over k'. We have surjections $K \otimes_k A \to K \otimes_{k'} A$ and $K \otimes_k M \to K \otimes_{k'} M$. If $\mathfrak{M}' \subset K \otimes_{k'} A$ is the canonical maximal ideal kernel $(K \otimes_{k'} A \to K)$, then the inverse images of $\mathfrak{M}'$ primary submodules of $K \otimes_{k'} M$ are $\mathfrak{M}$-primary submodules of $K \otimes_k M$. It thus suffices to assume K algebraic over k.

If K is separable algebraic over k, our chore is very easy since from 4.1 we already have an isomorphism $A/P^{n+1} \cong K \otimes_k A/\mathfrak{M}^{n+1}$. Thus

$$\begin{aligned}M/P^{n+1}M &\cong (A/P^{n+1}) \otimes_k M \\ &\cong (K \otimes_k A/\mathfrak{M}^{n+1}) \otimes_K K \otimes_k M \\ &= K \otimes_k M/\mathfrak{M}^{n+1}(K \otimes_k M),\end{aligned}$$

and the $\mathfrak{M}$-primary submodules of $K \otimes_k M$ correspond bijectively with P-primary submodules of M. Thus we obtain Prop. 1.6(c).

In the general case K algebraic over k, to show that every P-primary submodule L of M comes from an $\mathfrak{M}$-primary submodule of $K \otimes_k M$, it is necessary to use the notions of the associated primes and primary decomposition of the extended submodule $L^e = (K \otimes_k A)L$ of $K \otimes_k M$. The details are essentially identical to the argument in Section 3 for ideals, and we leave them to the reader. This will finally give Prop. 1.6(b). ∎

It is interesting to reconsider Prop. 1.2 in the case under discussion, which guarantees that any subcomodule of $\mathrm{Diff}^n_{A/k}(M, K(M))$ contains a *non-zero* A-linear map D^0: $M \to K(M)$ in $\mathrm{Diff}^0_{A/k}(M, K(M))$. If $L \subset M$ is a proper submodule, then by Nakayama's Lemma $L + PM \neq M$, hence L generates a proper subspace of $M/PM = K(M)$. Thus L will be contained in the kernel of certain non-zero maps $M \to M/PM$, obtained by following the canonical map $M \to M/PM$ by a non-zero projection operator $M/PM \to M/PM$ of K-vector spaces. If $M = A$, then $M/PM = K$ and nothing happens; all P-primary ideals are contained in P. But if $M \neq A$, P-primary submodules L need not be contained in PM. The first invariant of L is then the subspace of M/PM generated by L and if L corresponds to the zeros of the comodule of differential operators $\mathscr{E} \subset \mathrm{Diff}^n_{A/k}(M, K(M))$, then this invariant is detected by $\mathscr{E}^0$.

6. Some Computations

We assume in this section that our ring A is finitely generated over the field k, and $P \subset A$ is a prime ideal with residue field K separably generated over k. We will "compute" $\mathrm{Diff}^n_{A/k}(A, K)$, and in the process prove that in this case the map (0.1) discussed in the introduction is an isomorphism. That is, we prove Proposition 1.7.

First consider the polynomial ring $A = k[X_1 \cdots X_s]$. We have $K = k(\zeta_1 \cdots \zeta_s)$ where $\zeta_i = X_i \pmod P$. Also, $K \otimes_k A \cong K[X_1 \cdots X_s]$, $\mathfrak{M} = \mathrm{kernel}(K \otimes_k A \to K) = (X_1 - \zeta_1, ..., X_s - \zeta_s)$. Then

$$\begin{aligned}\mathrm{Diff}^n_{A/k}(A, K) &\cong \mathrm{Diff}^n_{K\otimes_k A/K}(K \otimes_k A, K)\\ &\cong \mathrm{Hom}_K(K[X_1 \cdots X_s]/\mathfrak{M}^{n+1}, K).\end{aligned} \tag{6.1}$$

A natural basis for $\mathrm{Diff}^n_{k[X_1 \cdots X_s]/k}(k[X_1 \cdots X_s], K)$ is thus provided by the dual basis of the monomials in the $X_i - \zeta_i$ of weight no greater than n. If the characteristic is 0, we can take as basis the partial derivative operators

$$\frac{1}{(i_1)! \cdots (i_s)!} \frac{\partial^I}{\partial X_1^{i_1} \cdots \partial X_s^{i_s}} (\zeta_1 \cdots \zeta_s), \qquad I = (i_1 \cdots i_s), \quad \sum i_j \leqslant n.$$

Next suppose $k \subset L \subset K$ is a subfield such that K is separable algebraic over L.

PROPOSITION 6.2. *The restriction map* $\mathrm{Diff}^n_{K/k}(K, K) \xrightarrow{\approx} \mathrm{Diff}^n_{L/k}(L, K)$ *is an isomorphism.*

Proof. This is Theorem 17 of [3], but we will deduce Proposition 6.2 directly from Proposition 4.1. We know

$$\mathrm{Diff}^n_{L/k}(L, K) \cong \mathrm{Diff}^n_{K\otimes_k L/K}(K \otimes_k L, K) \cong \mathrm{Hom}_K(K \otimes_k L/\mathfrak{M}_L^{n+1}, K)$$

where $\mathfrak{M}_L = \mathrm{kernel}(K \otimes_k L \to K)$. But now if we regard $K \otimes_k L$ as a ring with ground field $1 \otimes L$ and maximal ideal $\mathfrak{M}_L$, Prop. 4.1 gives an *isomorphism* $K \otimes_k L/\mathfrak{M}_L^{n+1} \xrightarrow{\approx} K \otimes_k L \otimes_L K/\mathfrak{M}_K^{n+1}$ where $K \otimes_k K \otimes_L K \cong K \otimes_k K$ and $\mathfrak{M}_K = \mathrm{kernel}(K \otimes_k K \to K)$. This isomorphism is K-linear relative to the action by $K \otimes 1$ on both sides, hence the K-duals are isomorphic, which is exactly Proposition 6.2. ($K \otimes_k L$ is not a local ring, as in Prop. 4.1, but this is unimportant since $\mathfrak{M}_L \subset K \otimes_k L$ is a maximal ideal.) ∎

Let us rewrite the map of (0.1)

$$\sum \mathrm{Diff}^i_{A/L}(A, K) \otimes_K \mathrm{Diff}^j_{K/k}(K, K) \to \mathrm{Diff}^n_{A/k}(A, K) \tag{6.3}$$

where now we suppose $X_1, \ldots, X_d \in k[X_1 \cdots X_s]$ gives a separating transcendence base for K over k and we replace $A = k[X_1 \cdots X_s]$ by the (partial) localization $k(X_1 \cdots X_d)[X_{d+1} \cdots X_s] = L[Y]$ where $L = k(X_1 \cdots X_d)$. Here we are using Proposition 2.2 to justify the localization, and Proposition 2.6 to justify the existence of the composition pairing above. But

$$\mathrm{Diff}^i_{A/L}(A, K) \cong \mathrm{Diff}^i_{L[Y]/L}(L[Y], K)$$

and

$$\begin{aligned}\mathrm{Diff}^j_{K/k}(K, K) &\cong \mathrm{Diff}^j_{L/k}(L, K)\\ &\cong \mathrm{Diff}_{k[X]/k}(k[X], K).\end{aligned}$$

Both are pure polynomial computations, which are described by (6.1). The right-hand side of (6.3) is also described by (6.1) and direct inspection will show (6.3) to be an isomorphism in this case.

The general ring of finite type over k can be written $A = k[x_1 \cdots x_s] = k[X_1 \cdots X_s]/J$, where J is some ideal in the polynomial ring. If $P \subset A$ is our prime, we can think of $J \subset P \subset k[X_1 \cdots X_s]$. Again, assume $x_1 \cdots x_d$ is a separating transcendence base for K over k, replace A by $k(x_1 \cdots x_d)[y_{d+1} \cdots y_s]$ which is a quotient of $k(X)[Y]$ and let $L = k(x_1 \cdots x_d)$.

Now, $\mathrm{Diff}^n_{A/k}(A, K) \subset \mathrm{Diff}^n_{k(X)[Y]/k}(k(X)[Y], K)$ is the subcoalgebra of operators which vanish on J, by Proposition 2.3. Similarly,

$$\mathrm{Diff}^i_{A/L}(A, K) \subset \mathrm{Diff}^i_{k(X)[Y]/k(X)}(k(X)[Y], K)$$

is the subcoalgebra of operators vanishing on J. Because (6.3) is an isomorphism when $A = k(X)[Y]$ is the polynomial ring, we get by restriction to subcoalgebras that (6.3) is an injection for general A of finite type over k.

Finally, to prove 6.3 surjective, we will use induction on n. If $n = 0$, 6.3 reduces to the identity $K \otimes_K K = K$. We also have for any n, $\mathrm{Diff}^n_{A/L}(A, K) = \mathrm{Diff}^n_{A/k}(A, K) \cap \mathrm{Hom}_L(A, K)$. This is clear from 1.1 since the $A \otimes_k A$ module structure on $\mathrm{Hom}_L(A, K) \subset \mathrm{Hom}_k(A, K)$ factors through $A \otimes_k A \to A \otimes_L A$. Thus for any n, the L-linear operators in $\mathrm{Diff}^n_{A/k}(A, K)$ are in the image of 6.3. Given any $D \in \mathrm{Diff}^n_{A/k}(A, K)$ we will prove a certain operator D' below is L-linear. It will also be clear by induction that $D - D'$ is in the image of 6.3, hence so is D.

Specifically, let

$$D' = D + \sum_{r=1}^{n} (-1)^r \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_r \leqslant d} D_{i_1 \cdots i_r} \circ [\cdots[[D, x_{i_1}], x_{i_2}] \cdots x_{i_r}] \tag{6.4}$$

where the

$$\begin{aligned} D_{i_1 \cdots i_r} &\in \mathrm{Diff}^r_{K/k}(K, K) \\ &= \mathrm{Diff}^r_{L/k}(L, k) \\ &= \mathrm{Diff}^r_{k[x_1 \cdots x_d]/k}(k[x_1 \cdots x_d], K) = \mathrm{Diff}^r_{K[x_1 \cdots x_d]/K}(K[x_1 \cdots x_d], K) \\ &= \mathrm{Hom}_K\left(\frac{K[x_1 \cdots x_d]}{\mathfrak{M}_L^{n+1}}, K\right) \end{aligned}$$

are the operators dual to the basis of monomials $(x_{i_1} - \zeta_{i_1}) \cdots (x_{i_r} - \zeta_{i_r})$. (Here, $\mathfrak{M}_L = (x_1 - \zeta_1, \ldots, x_d - \zeta_d) \subset K[x_1 \cdots x_d]$.) By induction, the terms $\pm D_{i_1 \cdots i_r} \circ [\cdots[D, x_{i_1}], \cdots x_{i_r}]$ which occur in $D - D'$ are all in the image of the triple composition

$$\sum_{i+j=n-r} \mathrm{Diff}^i_{A/k}(A, K) \otimes_K \mathrm{Diff}^j_{K/k}(K, K) \otimes \mathrm{Diff}^r_{K/k}(K, K) \to \mathrm{Diff}^n_{A/k}(A, K)$$

since $[\cdots[D, x_{i_1}] \cdots x_{i_r}] \in \mathrm{Diff}^{n-r}_{A/k}(A, K)$. By associating another way, this triple composition factors through 6.3, as desired.

It remains to prove that D' is L-linear, that is, we must prove

$$[D', f(x)/g(x)] = 0,$$

all $f(x)/g(x) \in k(x_1 \cdots x_d) = L$. Since $[D', f(x)/g(x)] = [D', f(x)] \circ (1/g(x)) - (f(x)/g(x)) \circ [D', g(x)] \circ (1/g(x))$ it suffices to prove $[D', f(x)] = 0$ for all polynomials $f(x) \in k[x_1 \cdots x_d]$. Since $[D', 1] = 0$ and $[D', p(x)\, q(x)] = [D', p(x)] \circ q(x) + p(x) \circ [D', q(x)]$, it suffices to prove that $[D', x_i] = 0$,

$1 \leqslant i \leqslant d$. The formula 6.4 for D' does not depend on the ordering of the $\{x_1, ..., x_d\}$ because of the equalities $D_{i_1 \cdots i_r} = D_{i_{\sigma(1)} \cdots i_{\sigma(r)}}$ and $[\cdots[D, x_{i_1}], ..., x_{i_r}] = [\cdots[D, x_{i_{\sigma(1)}}], ..., x_{i_{\sigma(r)}}]$ for any permutation σ of $\{1, ..., r\}$. Thus it suffices to prove $[D', x_d] = 0$. This will, in fact, be an easy computation given the following formula.

If $1 \leqslant i_1 \leqslant \cdots \leqslant i_r \leqslant d$ then

$$\begin{aligned} [D_{i_1 \cdots i_r}, x_d] &= \begin{cases} D_{i_1 \cdots i_{r-1}} & \text{if } i_r = d \\ 0 & \text{if } i_r < d \end{cases} \\ &\in \operatorname{Diff}^{r-1}_{K/k}(K, K) \\ &= \operatorname{Diff}^{r-1}_{K[x_1 \cdots x_d]/K}(K[x_1 \cdots x_d], K). \end{aligned} \tag{6.5}$$

(If $r = 1$, 6.5 means $[D_i, x_d] = 1$ if $i = d$, 0 if $i < d$.)

The proof of 6.5 is easy working in $\operatorname{Diff}^{r-1}_{K[x_1 \cdots x_d]/K}(K[x_1 \cdots x_d], K)$ since then $[D_{i_1 \cdots i_r}, x_d] = [D_{i_1 \cdots i_r}, x_d - \zeta_d] = D_{i_1 \cdots i_r} \circ (x_d - \zeta_d)$ and, by definition, $D_{i_1 \cdots i_r}$ is the operator which picks out the coefficient of $(x_{i_1} - \zeta_{i_1}) \cdots (x_{i_r} - \zeta_{i_r})$ in the Taylor series expansion of any polynomial about the point $(\zeta_1 \cdots \zeta_d)$.

Finally, we compute $[D', x_d]$ from 6.4 and 6.5 and the universal formula $[D_1 \circ D_2, a] = [D_1, a] \circ D_2 + D_1 \circ [D_2, a]$.

$$\begin{aligned} [D', x_d] &= [D, x_d] + \sum_{r=1}^{n} (-1)^r \sum_{i_1 \leqslant \cdots \leqslant i_r} [D_{i_1 \cdots i_r} \circ [\cdots[D, x_{i_1}] \cdots x_{i_r}], x_d] \\ &= 0 \end{aligned}$$

since all terms cancel in pairs using

$$\begin{aligned} &\sum_{i_1 \leqslant \cdots \leqslant i_r} [D_{i_1 \cdots i_r} \circ [\cdots[D, x_{i_1}] \cdots x_{i_r}], x_d] \\ &\quad = \sum_{i_1 \leqslant \cdots \leqslant i_{r-1}} D_{i_1 \cdots i_{r-1}} \circ [[\cdots[D, x_{i_1}], \cdots x_{i_{r-1}}], x_d] \\ &\qquad + \sum_{i_1 \leqslant \cdots \leqslant i_r} D_{i_1 \cdots i_r} \circ [[\cdots[D, x_{i_1}] \cdots x_{i_r}], x_d] \quad \blacksquare. \end{aligned}$$

We summarize our computation of $\operatorname{Diff}^n_{A/k}(A, K)$, where $A = k(x)[y]$, $P \subset A$ a zero dimensional prime over $L = k(x)$. We have the isomorphism (6.3):

$$\sum \operatorname{Diff}^i_{A/k(x)}(A, K) \otimes_K \operatorname{Diff}^j_{K/k}(K, K) \xrightarrow{\sim} \operatorname{Diff}^n_{A/k}(A, K).$$

We have

$$\operatorname{Diff}^j_{K/k}(K, K) \cong \operatorname{Hom}_K(K[x]/\mathfrak{M}^{n+1}, K)$$

where $\mathfrak{M}$ here refers to the ideal $(x - \zeta) \subset K[x]$. The operators are the partial derivatives with respect to the variables x, at least in characteristic 0. We have

$$\begin{aligned}\mathrm{Diff}^i_{A/k(x)}(A, K) &\cong \mathrm{Hom}_K(K[y]/(\mathfrak{M}')^{n+1}, K) \\ &\cong \mathrm{Hom}_{k(x)}(A/P^{n+1}, k(x)).\end{aligned}$$

The second identification is based on our trace discussion in Section 4, and the isomorphism 4.1, $A/P^{n+1} \cong K \otimes_{k(x)} A/(\mathfrak{M}')^{n+1}$. If we write $A = k(X)[Y]/J$, $J \subset P \subset k(X)[Y]$, then A/P^{n+1} becomes $k(X)[Y]/J + P^{n+1}$.

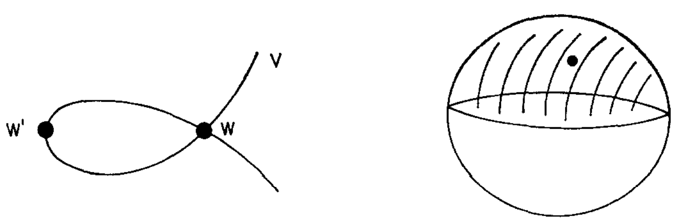
If A is interpreted as a coordinate ring of an affine variety V and $P \subset A$ is interpreted as defining an irreducible subvariety $W \subset V$, then the decomposition (6.3) is analogous to the decomposition of the symmetric powers of a sum of two vector bundles. The tangent bundle of V restricted to W breaks into a component tangent to W, corresponding to the $\mathrm{Diff}^j_{K/k}(K, K)$ in (6.3), and a component normal to W in V, corresponding to the $\mathrm{Diff}^i_{A/k(x)}(A, K)$. The tangent component is not too interesting. For example, a basis over K of $\mathrm{Diff}^j_{K/k}(K, K)$ always corresponds to the monomials of weight no greater than j in d-variables where d is the dimension of W. This is because generically W really is a manifold.

On the other hand, the normal component depends on the placement of W in V and is especially interesting if W sits in the singular set of V. For example, if $A = k[X, Y]/(Y^2 - X^2 - X^3)$, $P = (X, Y)$, then W is a point on the curve V, but

$$\dim_k(\mathrm{Diff}^1_{A/k}(A, k)/\mathrm{Diff}^0_{A/k}(A, k)) = 2$$

and

$$\dim_k(\mathrm{Diff}^2_{A/k}(A, k)/\mathrm{Diff}^1_{A/k}(A, k)) = 2.$$



If W' is a simple point, the corresponding dimensions are 1 and 1. A simple point on a surface would give dimensions 2 and 3.

The general philosophy here is just as in the study of singularities. If the first order normal term $\mathrm{Diff}^1_{A/k(x)}(A, K)$ is well behaved, then W will sit non-singularly in V, at least generically. If W is a singular subvariety, more delicate classification will result from the higher terms $\mathrm{Diff}^i_{A/k(x)}(A, K)$. These invariants essentially come from the local ring of W in V.

It is perhaps instructive to look at the K-duals of the two sides of 6.3. In fact, since it was easy to prove 6.3 injective, one approach to surjectivity would be to simply show the K-duals of both sides have the same dimension over K.

Write $A = k[x_i, x_{d+j}]/(f(x_i, x_{d+j}))$, where the denominator denotes generators of the ideal of relations. Then $\mathrm{Diff}^n_{A/k}(A, K) = \mathrm{Diff}^n_{K\otimes_k A/K}(K \otimes_k A, K)$ is dual to the finite dimensional K-algebra.

$$K[x_i, x_{d+j}]/(f(x_i, x_{d+j})) + \mathfrak{M}^{n+1} \qquad (*)$$

where $\mathfrak{M} = (x_i - \zeta_i, x_{d+j} - \zeta_{d+j})$. On the other hand, the left hand side of 6.3 can be shown to be another expression for $\mathrm{Diff}^n_{(K\otimes_L A)\otimes_K(K\otimes_k L)/k}((K \otimes_L A) \otimes_k (K \otimes_k L), K)$ where now $L = k[x_i]$ since we haven't localized A. This identification can be made via the filtration argument used to prove Proposition 1.4(a) at the end of Section 2. We will skip the details. A proof can also be found in [4, Theorem 13.19]. In any event, since $K \otimes_k L = K[x_i]$ and $K \otimes_L A = K[x_{d+j}]/(f(\zeta_i, x_{d+j}))$, we see that the left hand side of 6.3 is dual to

$$K[x_i, x_{d+j}]/(f(\zeta_i, x_{d+j})) + \mathfrak{M}^{n+1} \qquad (**)$$

At first glance, it would seem not difficult to relate (*) and (**). Perhaps these K-algebras have isomorphic associated graded algebras. However, we did not succeed in our attempt to find a simple proof that 6.3 is an isomorphism by relating (*) and (**). In fact, it seems that the map 6.3, composition of operators, preserving coproducts (in a certain twisted sense; see the introduction) occurs more naturally than any relation between the K-algebras (*) and (**).

In [4, Sects. 8 and 13], Sweedler proves many other interesting results about differential operators. His method is roughly to represent the functor $\mathrm{Diff}^n_{A/k}(A, M) = \mathrm{Hom}_A(J_n(A), M)$ by constructing a universal operator $j_n\colon A \to J_n(A)$. Then properties of differential operators are deduced from properties of $J_n(A)$. For example, Sweedler's Theorem 13.12 gives by this method a generalization of our Proposition 6.2 above.

7. Further Remarks

In this section we make some suggestions which are not exactly conjectures, but rather further projects which would complement our results.

7.1. The inseparable case should be treated much more carefully. Aside from a couple of vague remarks, we have given no results other than the imprecise 1.5(a), which holds in general. What is wanted is an analogue of 1.5(b), or a more general statement which includes both separable and purely inseparable cases.

7.2. We have assumed the Noetherian ring A contains a field. Without some modification of our method this seems necessary. For example, if $A = \mathbb{Z}$, $P = (p)$, then $\mathrm{Diff}_{\mathbb{Z}}{}^n(\mathbb{Z}, \mathbb{Z}/p) = \mathrm{Diff}_{\mathbb{Z}}{}^0(\mathbb{Z}, \mathbb{Z}/p) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/p) = \mathbb{Z}/p$, and the primary ideals $(p^n) \subset \mathbb{Z}$ cannot be detected. Note that in this case, at least, the P-primary ideals can be detected using the associated graded ring $G\mathbb{Z} = \bigoplus_{i=0} p^i\mathbb{Z}/p^{i+1}\mathbb{Z}$, which contains the field $\mathbb{Z}/p$. In fact, $G\mathbb{Z} \cong \mathbb{Z}/p[x]$, the polynomial ring. An integer n has an expansion

$$n = a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k = f(x)|_{x=p},$$

$a_i \in \mathbb{Z}/p$, and the coefficients a_i correspond to evaluating higher order differential operators on $G\mathbb{Z}$.

7.3. In general, what application can be made of filtrations, completions, and graded rings and modules toward better understanding differential operators and primary ideals and submodules?

7.4. The primary ideals in non-Noetherian rings might be accessible in some cases. First, a P-primary ideal need contain no power P^{n+1}. Thus it would be necessary to allow differential operators of all orders. Secondly, even the operators of finite degree n need not define a finite dimensional coalgebra and, in fact, operators might not even have diagonal formulas in general. In such a case about the best one could hope for would be a true "converse" of Proposition 1.3, characterizing primary ideals as zeros of subspaces of differential operators, closed under the bracket operation.

7.5. It would be very natural to extend our results from modules over rings to sheaves of modules over ringed spaces.

7.6. In differential topology, differential operators occur naturally, mapping sections of one vector bundle to sections of another over a base manifold V. The ring A is the ring of all smooth functions on V, which is not noetherian, but which is still somewhat manageable. Associated to a pair of vector bundles over V are certain jet bundles, whose sections correspond to differential operators of given order. These jet bundles and their duals play a role in the study and classification of singularities. Our results suggest that there might be good reason to consider various subbundles of these jet bundles defined by subcomodule type conditions. In other words, in the study of primary ideals one does not restrict attention to powers of a maximal ideal (or, more generally, symbolic powers of a prime ideal).

7.7. It is now known that every closed compact smooth manifold is diffeomorphic to a non-singular real algebraic variety. This result seems to me to justify studying carefully the purely algebraic theory of differential operators, say in connection with index and Riemann–Roch type problems on real algebraic varieties.

Now, I do not think real algebraic geometry, or better, semi-algebraic geometry, where inequalities are allowed, has really gotten off the ground. There are some very good theorems in the literature, but part of the problem has been confusion about where pure algebra ends and topology begins (limits, continuity, etc.). Consider, for example, the question of what category of finitely generated rings A over a real closed field R is natural for studying semi-algebraic sets in affine space. In the reduced case, we want A to be a ring of functions and an obvious necessary condition is the following. If $\sum a_i^2 = 0 \in A$, then $a_j = 0$, all j. The real Nullstellensatz says, in fact, that this condition is also sufficient, in the sense that if $A = R[X_1 \cdots X_s]/I$, then A is naturally a ring of functions on the zero set of I in R^s.

However, it is not as well understood what non-reduced rings of finite type arise naturally. It seems clear that what one should do is study the same sort of intersection problems, fibres of morphisms, specializations, and so on, in which nilpotent elements arise in classical algebraic geometry. *The relations of differential operators with real functions and positivity should play a key role.* The subtlety here is that over an algebraically closed field, *all* algebras of finite type occur for geometric reasons, but one knows over a real closed field there are natural restrictions—witness the reduced case. Here is may candidate for the basic geometric restriction on rings over a real field. If $(\sum a_i^2)x = 0$, then $a_j^2 x = 0$, all j.

All this will hopefully be justified eventually in a project on foundations of semi-algebraic geometry. I dwell on it here only because these considerations are what originally led me to worry about what a primary ideal really was and then to the results formulated in this paper.

7.8. Finally, for a complete change of pace, it would seem interesting and not necessarily very difficult to extend much of the theory of differential operators to modules over non-commutative algebras. An example which might have some topological significance would be the Steenrod algebra $A(p)$, and the cohomology modules $H^*(X, \mathbb{Z}/p)$, X a space.

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